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A note on the validity of the eikonal approximation

Yang Yong-Xu† and Li Qing-Run‡

† Department of Physics, Guangxi Normal University, Guilin, People's Republic of China
‡ CCAST (World Laboratory), Beijing, People's Republic of China and Institute of High Energy Physics, Academia Sinica, Beijing, People's Republic of China

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Abstract. It has been suggested that for large incident wavenumbers and small scattering angles the eikonal approximation remains excellent for strong-coupling situations. We show that this conclusion has limitations, and the validity of the eikonal approximation depends on the sharpness of the edge of the potentials.

The eikonal approximation has been used extensively in studies of a variety of intermediate- and high-energy particle collision processes. In nuclear physics, especially, this approximation is applied very frequently and extensively. In the work of Glauber [1] the criteria for the validity of the eikonal approximation are given to be $ka \gg 1$ and $|V_0|/E \ll 1$. In a series of works [2–5] Byron *et al* have made a systematic study of the range of validity of the eikonal approximation. One of their conclusions is that for large incident wavenumbers ($ka \gg 1$) and strong coupling ($|V_0|/E \gg 1$) the eikonal approximation remains excellent for small scattering angles [3, 4]. This is in contrast to the usual criteria of validity of the eikonal approximation [1].

In this paper we shall show that the above-mentioned conclusion has a limitation, namely it is only valid for potentials without a sharp edge, but not valid for potentials with a sharp edge.

Let us consider the scattering of a particle of mass m by a central potential $V(r)$ of range a. The eikonal scattering amplitude is

$$f_{\mathrm{E}}(q) = \frac{k}{\mathrm{i}} \int_0^\infty \mathrm{d}b\, bJ_0(qb)(\mathrm{e}^{\mathrm{i}\chi(b)} - 1) \tag{1}$$

where

$$\chi(b) = -\frac{1}{2k} \int_{-\infty}^{\infty} U(b, z)\, \mathrm{d}z \tag{2}$$

and k is the incident wavenumber, $U = 2mV(r)/\hbar^2$ is the reduced potential and q is the magnitude of the momentum transfer. The amplitude in (1) is obtained on the assumption that the conditions $ka \gg 1$ and $|V_0|/E \ll 1$ are satisfied, where $|V_0|$ is a typical strength of the potential and E is the incident energy of the particle. Thus, these inequalities are usually considered to be the criteria of validity of the eikonal approximation.

Byron *et al* have systematically studied [2–5] the validity of the eikonal approximation for a variety of cases. They found that for strong coupling ($|V_0|/E \gg 1$) and large incident wavenumbers ($ka \gg 1$) the eikonal approximation is excellent for small-angle scattering [3, 4]. In [3, 5] they dealt with a Yukawa potential of the form $U(r) = U_0\,\mathrm{e}^{-r/a}/r$ and a polarization potential of the form $U(r) = U_0/(r^2+d^2)^2$ and took $U_0 = -250$, $k = 5$, $a = d = 1$ (unit of length) as the parameters for a strong-coupling case ($|V_0|/E = |U_0|/k^2 = 10$ and $ka = 5$). The differential cross sections for the Yukawa potential are shown in figure 1. (The results in figure 1 are calculated by us and are in agreement with those in figure 9.8 of [5], where the curves are only given for angles $\theta < 140°$). We see that for angles $\theta < 10°$ the agreement between the eikonal and exact results is indeed excellent. For the polarization potential, the situation is similar to that for the Yukawa potential (see figure 1 of [3]). These results led to the foregoing conclusion. However, we shall show that it is not the same if we turn our attention to some other potentials. Let us consider Gaussian potentials of the form

$$U(r) = U_0 \exp(-r^2/a_g^2) \tag{3}$$

and square-well potentials of the form

$$U(r) = \begin{cases} U_0 & r < a_s \\ 0 & r > a_s. \end{cases} \tag{4}$$

Corresponding to $a = 1$ for the Yukawa potential, a_g and a_s should be taken as 2.0 and 3.15, respectively so that the same values of RMS, $\langle r^2\rangle^{1/2} = \int U(r) r^2\,\mathrm{d}^3r / \int U(r)\,\mathrm{d}^3r$, can be obtained for the Yukawa, Gaussian and square-well potentials. For strong coupling and large wavenumbers, we also take $U_0 = -250$, $k = 5$ for comparison with the results of [3]. The differential cross sections for the eikonal and exact calculations are plotted in figures 2 and 3. From these figures it can be seen that for the Gaussian potential the eikonal curve at small angles deviates from the exact curve to some extent

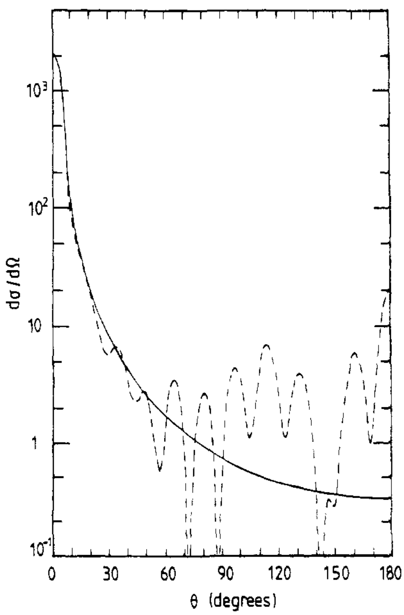

Figure 1. The differential cross section for the Yukawa potential with $U_0 = -250$, $a = 1$ (unit of length) and $k = 5$. The solid curve shows the eikonal result, the broken curve gives the exact result.

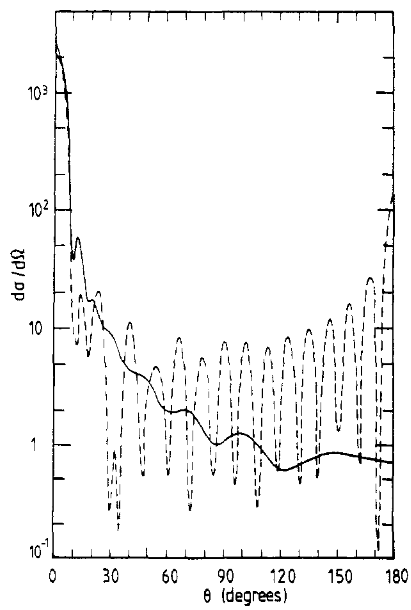

Figure 2. The differential cross section for the Gaussian potential of the form (3), with $U_0 = -250$, $a_g = 2.0$ and $k = 5$. The solid curve shows the eikonal result, the broken curve gives the exact result.

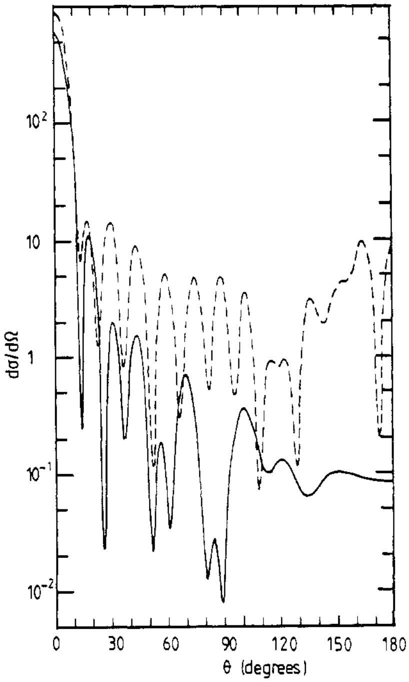

Figure 3. The differential cross section for the square well potential of the form (4), with $U_0 = -250$, $a_s = 3.15$ and $k = 5$. The solid curve shows the eikonal result, the broken curve gives the exact result.

and for the square-well potential the deviation is about 50% at $\theta = 0°$. In order to have more insight into this point, let us consider the cases with other potential parameters. We take $k = 2$ and $U_0 = -20$, which are used in [2] for strong coupling. The differential cross sections for the Yukawa, Gaussian and square-well potentials are shown in figure 4. We see again that the eikonal approximation yields very good results at small angles for the Yukawa potential, but for the square-well potential the agreement between the eikonal and exact results is lost. Let us now study the variation of the differential cross section at $\theta = 0°$ as a function of U_0. We still take $k = 5$. The differential cross sections at $\theta = 0°$ for the three types of potentials considered above are plotted against U_0 in Figure 5. We obtain an interesting result: both the eikonal and exact curves have remarkable oscillating behaviour for the square-well potential. This can be understood for the eikonal case. For the square-well potential in (4), the eikonal amplitude is

$$f_E(q) = \frac{k}{i} \int_0^{a_s} db\, b J_0(qb) \left[\exp\left(-i \frac{U_0}{k} \sqrt{a_s^2 - b^2} \right) - 1 \right]. \tag{5}$$

For $\theta = 0°$, we have

$$f_E(0) = \frac{k}{i} \int_0^{a_s} db\, b \left[\exp\left(-i \frac{U_0}{k} \sqrt{a_s^2 - b^2} \right) - 1 \right] \tag{6}$$

$$= \left[\frac{k^2}{U_0} \left(a_s \cos \frac{a_s U_0}{k} - \frac{k}{U_0} \sin \frac{a_s U_0}{k} \right) \right]$$

$$+ i \left[\tfrac{1}{2} k a_s^2 + \frac{k^2}{U_0} \left(\frac{k}{U_0} - \frac{k}{U_0} \cos \frac{a_s U_0}{k} - a_s \sin \frac{a_s U_0}{k} \right) \right]. \tag{7}$$

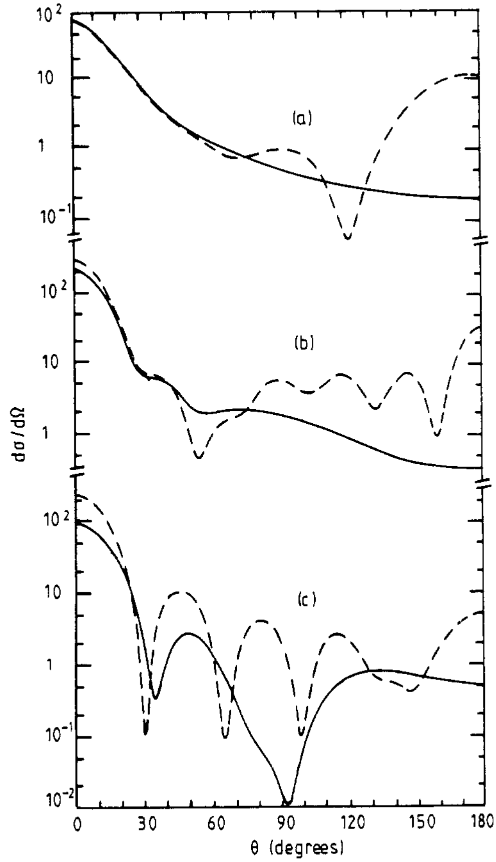

Figure 4. The differential cross sections for: (*a*) the Yukawa potential with $U_0 = -20$, $a = 1$ and $k = 2$; (*b*) the Gaussian potential with $U_0 = -20$, $a_g = 2.0$ and $k = 2$; (*c*) the square-well potential with $U_0 = -20$, $a_s = 3.15$ and $k = 2$. The solid curves show the eikonal results, the broken curves give the exact results.

Thus the differential cross section at $\theta = 0°$ is

$$\left(\frac{d\sigma}{d\Omega}\right)_{\theta=0°} = |f_E(0)|^2$$

$$= \tfrac{1}{4}k^2a_s^4 + 2\frac{k^4a_s^2}{U_0^2} + 2\frac{k^6}{U_0^4} - \frac{k^4}{U_0^2}\left(a_s^2 + 2\frac{k^2}{U_0^2}\right)\cos\frac{a_sU_0}{k}$$

$$-\frac{k^3a_s}{U_0}\left(a_s^2 + 2\frac{k^2}{U_0^2}\right)\sin\frac{a_sU_0}{k}. \tag{8}$$

Equations (7) and (8) cannot be used for $U_0 = 0$. Equation (8) shows that the differential cross sections at $\theta = 0°$ have oscillation behaviour induced by the trigonometric functions and approach a constant, $\frac{1}{4}k^2a_s^4$, as $|U_0|$ becomes large. The behaviour of the exact differential cross sections at $\theta = 0°$ for the square-well potential might not be studied analytically, although the phase shifts can be obtained analytically. In this case, however, the exact amplitudes at $\theta = 0°$, $f(0) = (1/2ki)\sum_l(2l+1)[\exp(2i\delta_l) - 1]$, can be calculated to high accuracy. Figure 5 also shows that for the square-well potential the eikonal approximation can give very good results only around the intersection points of the two curves and might have errors of about 100% at some other places.

The results presented above seem to suggest that for strong coupling and small-angle scattering, the validity of the eikonal approximation depends on the sharpness of the

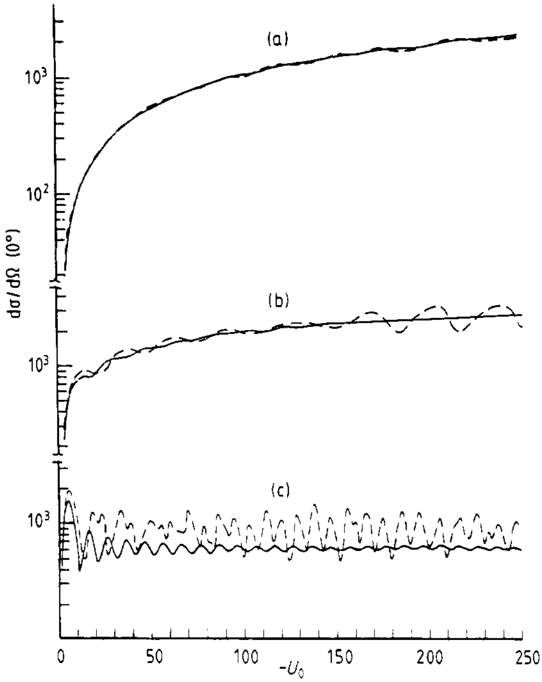

Figure 5. The differential cross section at $\theta = 0°$ for $k = 5$ as functions of U_0 for three types of the potential: (a) Yukawa, $a = 1$; (b) Gaussian, $a_g = 2.0$; (c) square-well, $a_s = 3.15$. The solid curves show the eikonal results, the broken curves give the exact results.

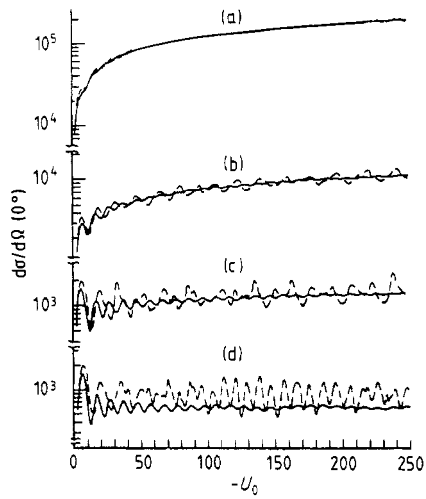

Figure 6. The differential cross section at $\theta = 0°$ for $k = 5$ as functions of U_0 for the Woods-Saxon potential with four different thickness parameters: (a) $t/a_w = 0.5$; (b) $t/a_w = 0.2$; (c) $t/a_w = 0.05$; (d) $t/a_w = 0.0006$. The solid curves show the eikonal results, the broken curves give the exact results.

edges of the potentials. In order to gain an insight onto this question, we deal with the Woods-Saxon potentials of the form $U(r) = U_0/\{1 + \exp[(r - a_w)/t]\}$ with four different thickness parameters $t/a_w = 0.5, 0.2, 0.05, 0.0006$ and $a_w = 3.15$. The differential cross sections at $\theta = 0°$ for $k = 5$ are given in figure 6 as a function of U_0. Obviously, the validity of the eikonal approximation for small angles depends on the thickness parameters of the potentials. Note that the Woods-Saxon potential with $t/a_w = 0.0006$ is virtually indistinguishable from the square-well potential.

For complex potentials, the situation is similar to that of real potentials. Taking the reduced strength of the potential as $U_0(1 + \mathrm{i})$, the differential cross sections at $\theta = 0°$ for Yukawa, Gaussian and square-well potential are shown as functions of U_0 in figure 7. In contrast to the real potential case, the oscillating behaviour disappears. This is due to the fact that a damping factor $\exp(a_s U_0/k)$ ($U_0 < 0$) which comes from the imaginary part of the potential is multiplied to the trigonometric functions in (8). From the figure, one can see that for the complex square-well potential considered here the eikonal differential cross section at $\theta = 0°$ might have errors of about 40%.

Finally, we present the results for intermediate coupling ($ka \gg 1$ and $|V_0|/E \leqslant 1$). In [2], one concluded that for intermediate coupling the eikonal approximation can very well reproduce the exact result for all scattering angles with Yukawa-type potential. We show that, similar to the strong-coupling situation, it would be different if one uses potentials with sharp edges. Our results for intermediate-coupling cases are

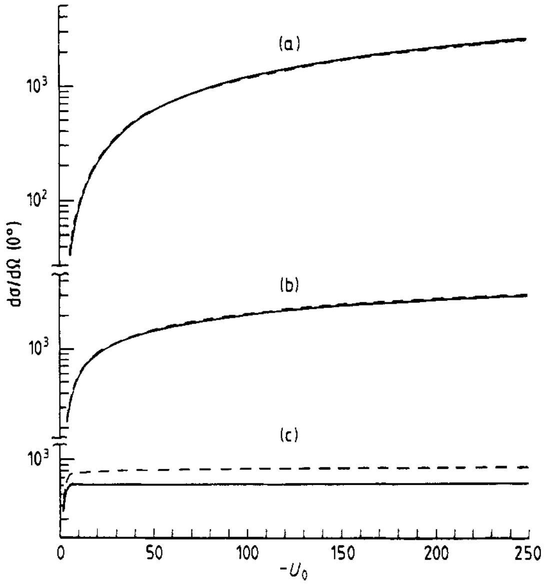

Figure 7. The differential cross section at $\theta = 0°$ for $k = 5$ as functions of U_0 for three types of the complex potential: (*a*) Yukawa, $a = 1$; (*b*) Gaussian, $a_g = 2.0$; (*c*) Square well, $a_s = 3.15$. The solid curves show the eikonal results, the broken curves give the exact results.

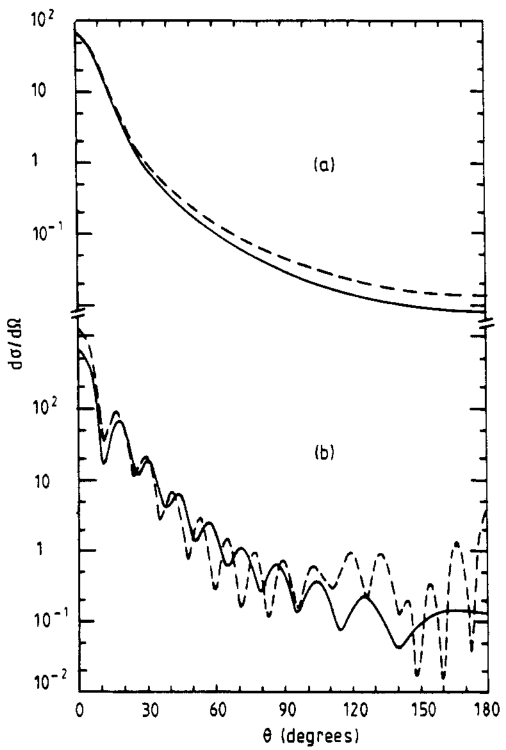

Figure 8. The differential cross sections for (*a*) the Yukawa potential with $U_0 = -10$, $a = 1$ and $k = 5$; (*b*) the square-well potential with $U_0 = -10$, $a_s = 3.15$ and $k = 5$. The solid curves show the eikonal results, the broken curves give the exact results.

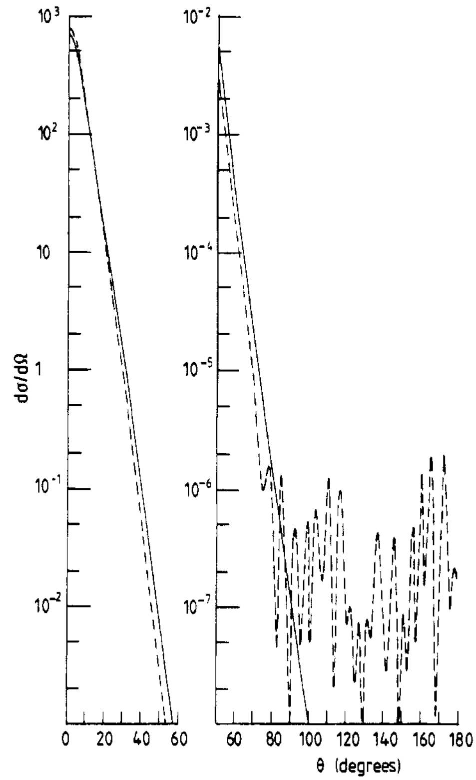

Figure 9. The differential cross section for the Gaussian potential with $U_0 = -10$, $a_g = 2.0$ and $k = 5$. The solid curve shows the eikonal result, the broken curve gives the exact result.

displayed in figures 8 and 9. Also from figure 5, one can see that in the region $-U_0 \approx 10$–30 (intermediate-coupling cases) the eikonal approximation might have errors of about 100% for the forward angle. Intermediate coupling might be met in the interactions of strongly absorbed particles or heavy ions with the heavier nuclei.

In conclusion, for strong coupling and large incident wavenumber or intermediate coupling and large incident wavenumber, the eikonal approximation might not yield good results for small angles. In these cases, the validity of the eikonal approximation in the small-angle region depends on sharpness of the edges of the potentials.

References

[1] Glauber R J 1959 *Lectures in Theoretical Physics* ed W E Brittin (New York: Interscience) vol 1, p 315
[2] Byron F W Jr, Joachain C J and Mund E H 1973 *Phys. Rev.* D **8** 2622
[3] Byron F W Jr, Joachain C J and Mund E H 1975 *Phys. Rev.* D **11** 1662
[4] Byron F W Jr, Joachain C J and Mund E H 1979 *Phys. Rev.* C **20** 2325
[5] Joachain C J 1975 *Quantum Collision Theory* (Amsterdam: North-Holland) ch 9